1st International Conference "From Scientific Computing to Computational Engineering"
1st IC-SCCE
Athens, 8-10 September, 2004

NUMERICAL SOLUTION OF THE BOUSSINESQ EQUATION USING SPECTRAL METHODS AND STABILITY OF SOLITARY WAVE PROPAGATION

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Keywords: Spectral methods, Boussinesq equation, nonlinear waves.

Abstract. *We study numerically the propagation and stability properties of solitary waves (solitons) of the Boussinesq equation in one space dimension, using a combination of finite differences in time and spectral methods in space. Our schemes follow very accurately these solutions, which are given by simple closed formulas and are known to be stable under small perturbations, for small enough velocities. Studying the interaction of two such solitons, we determine in the velocity parameter plane a sharp curve beyond which they become unstable. This is achieved by applying a precise criterion, which predicts when the observed amplitude growth of the waves is caused by a dynamical instability rather than the accumulation of numerical errors.*

1 INTRODUCTION

Discretization using finite differences in time and spectral methods in space has proved to be very useful in solving numerically non-linear partial differential equations (PDE) describing wave propagation. In recent studies, we have solved the Korteweg de Vries (KdV) equation[1] and the generalized KdV equation using such combined schemes and have analyzed efficiently unidirectional solitary wave propagation in one dimension[2,3]. We have determined that these waves interact elastically in all cases and have computed detailed stability thresholds in the space of physical parameters of the problem. In particular we have obtained specific velocity values beyond which solitary waves break down due to dynamical and not computational instabilities of the equations. It is particularly interesting that exact, analytical expressions of these solitary waves known from the KdV equation continue to exist in remarkably similar form for the generalized KdV as well and for wide range of parameter values. For parameters where such analytical expressions are not available reduction to an ordinary differential equation (ODE) and standard phase plane analysis can still be used to obtain the solitary wave numerically as a separatrix solution of the ODE.

In this study we apply a combination of spectral methods and finite differences to another well-known non-linear PDE describing water waves, called the Boussinesq equation. This equation admits bidirectional wave propagation, has closed form solitary wave solutions and like the KdV is completely integrable in one space dimension. These solutions are called solitons and are known to exist in arbitrary number and interact completely elastically. We numerically follow these interactions and investigate their stability properties by varying the velocity parameter of the waves which appears in their analytical form. Our future purpose is to compare these results with those of other numerical schemes, which use e.g. finite differences in both time and space.

2 MATHEMATICAL FORMULATION

Let us consider the well-known Boussinesq equation[1]

$$\frac{\partial^2 u}{\partial t^2}-\frac{\partial^2 u}{\partial x^2}+3\frac{\partial^2 u^2}{\partial x^2}+\frac{\partial^4 u}{\partial x^4}=0 \tag{1}$$

Looking for travelling wave solutions of equation (1) of the form

$$u(x,t)=f(x-x_1-ct) \tag{2}$$

we obtain an ODE which can be easily integrated twice. Setting the two integration constants equal to zero it is not difficult to show that this ODE has the solution

$$u(x,t)=2b^2 sech^2\left(b(x-x_1-ct)\right) \tag{3}$$

representing a solitary wave, where $c=\pm\sqrt{1-4b^2}$ is the propagation speed and b, x_1 arbitrary constants determining the height and the position of the maximum height of the wave, respectively. From the form of c it is apparent that the solution can propagate in either direction (left or right).

The initial condition we use to numerically solve equation (1) can thus be extracted from the above relation for $t=0$. We should also mention that, in order to have the wave solution (2), parameter b must satisfy the relation $|b|<0.5$. Moreover, the maximum of the wave, $2b^2$, occurs at the point $x=x_1+ct$.

3 NUMERICAL METHOD

The numerical scheme used in the current study is the same as the one employed in[2,3] and is based on a combination of finite differences and a Fourier pseudospectral method[4]. In order to demonstrate the application of our algorithm we first describe it on the Boussinesq equation (1) with the initial condition given by equation (3).

The time derivative in equation (1) is discretised using a finite difference approximation, in terms of central differences

$$u^{n+1}=2u^n-u^{n-1}+(\Delta t)^2\left(u^n_{xx}-3\left(u^2\right)^n_{xx}-u^n_{xxxx}\right)=0 \tag{4}$$

According to the pseudospectral method, we introduce the approximate solution

$$u(x,t)=\sum_{k=0}^{N}a_k(t)\Phi_k(t) \tag{5}$$

where $\Phi_k(x)=e^{ikt}$ are the Fourier exponentials, and $a_k(t)$ are coefficients to be determined, for $k=0,1,\dots,N$.

The steps used to advance the solution from time step n to $n+1$ are[4]:

(i) Given $u^n_j=u(x_j,t_n)$ evaluate $a^n_k=a_k(t_n)$ from equation (5).

(ii) Given a^n_k evaluate the derivatives e.g. $\left[\frac{\partial^2 u}{\partial x^2}\right]^n_j$ from equation (5).

(iii) Evaluate the nonlinear terms e.g. $u^n_j\left[\frac{\partial u}{\partial x}\right]^n_j$.

(iv) Evaluate u^{n+1}_j from equation (4), at $x=x_j$, $t=t_{n+1}$

Step (i) is the transformation from physical space to spectral space. This transformation is achieved by using a Fast Fourier Transform (FFT) described in[5,6] with a number of operations $(5/2)N\log_2 N$ (N being the number of polynomials), in contrast to the $2N^2$ operations required for a matrix-vector multiplication. Step (ii) occurs in spectral space and the evaluation of the nonlinear term in step (iii) is in physical space, thus avoiding the expensive multiplication of all coefficients in the expansions of the form (5). Step (iv) occurs again in physical space.

The accuracy of our numerical scheme for the time variable t is $O((\Delta t)^2)$, due to central differences, while for the space variable x, where we use the pseudospectral method, the errors are $O(e^{-qN})$, where q is a constant[4]. Numerical calculations were carried out for various choices of in (5), $N = 128$, 256, 512 and 1024 and time steps $\Delta t = 0.0001$ to 0.002, while the spatial step was chosen to be $\Delta x = 1$.

3.1 Stability Criterion

Our scope, is to examine by numerical means, whether the values of the parameter b in the solitary wave solution (3) of our Bousinesq equation (1), affects their shape and their stability under evolution. By the term 'stable' we mean that a wave solution retains its initial profile under small perturbations, albeit with some smaller oscillations present as radiation waves, due to unavoidable numerical errors produced under time evolution.

Thus, in order to check stability, one way is to track the residual of the solution in time. For the case of Boussinesq, for example, if $\bar{u}$ is an exact solution of equation (1) it will satisfy

$$\bar{u}_{tt} - \bar{u}_{xx} + 3\bar{u}^2_{xx} + \bar{u}_{xxxx} = 0 \tag{6}$$

If the approximate solution (5), computed numerically, is substituted into (1) it will not, of course, give zero. Thus we write for it

$$u_{tt} - u_{xx} + 3u^2{}_{xx} + u_{xxxx} = R \tag{7}$$

where R is called the residual of the equation. It is expected that R is a continuous function of x and t and if N is sufficiently large then, in principle, the coefficients $a_k(t)$ can be chosen so that R is as small as we wish over the computational domain. In our case we evaluate $R = R_i$ at each x_i, $i = 1, 2, \ldots, N$ point and at specific time moments t_n.

Due to the fact that the wave solutions are computed for sufficiently large values of N (128 to 1024), the spatial error of the pseudospectral method is negligible, in agreement with the $O(e^{-qN})$ estimate mentioned above. The maximum absolute residual, which we refer to as the error $E = \max_i |R_i|$, will increase due to the central differencing in time, but cannot be greater than $O((\Delta t)^2)$. Several tests have been made for the wave solution (3) of the Boussinesq verifying that for various values of N (128 to 1024) and time step $\Delta t = 0.0001$ to 0.02, $E < (\Delta t)^2$ at least for a time interval of 1000 time units.

Therefore, a practical way to verify that a wave solution is stable is to check if the error remains, for long times, less than $O((\Delta t)^2)$. If E increases above this value already from the outset, oscillations will soon grow and become unbounded after relatively short times, not only because of the numerical scheme, but also due to the nonlinear nature of the equations, suggesting that the initial wave solution has become unstable. *This is also supported by the fact that blowup occurs nearly at the same times, irrespective of the values of the Δx and Δt step sizes used in the numerical scheme.*

4 RESULTS

In this section we investigate numerically the behavior of one or two waves for the Boussinesq equation (1) for different values of the parameter b.

4.1 One wave

We begin the investigation of the Boussinesq equation by taking as initial condition the solitary wave (3) at $t = 0$ with $b = 0.2$, $x_1 = 30$. We observe that our wave moves along the spatial direction retaining its initial profile for a very long time interval, at least for $t = 2.5\times10^6$ time units (see Figure 1). The propagation of the wave is pictured in Figure 1 for $dt = 0.001$ and N =128. In order to study the effect of the parameter b, we have performed calculations checking the error as described in section 3.1. As b increases, the stability of the wave propagating in time breaks down and for values of $|b|$ close to 0.5 the wave blows up. In fact, the maximum value of b for stability is nearly 0.4. If b exceeds this value the wave blows up after only 250 time units.

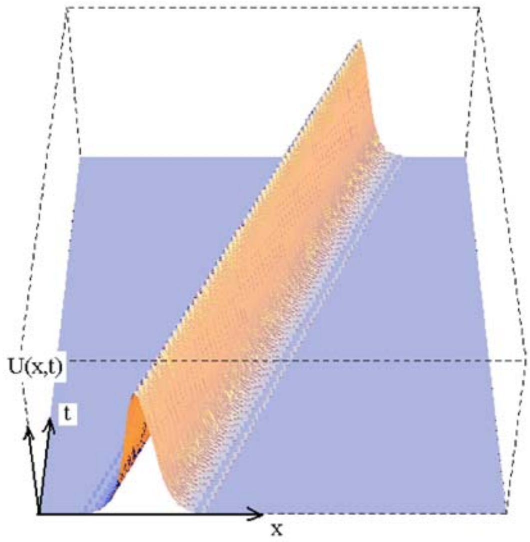

Figure 1. Propagation of one wave.

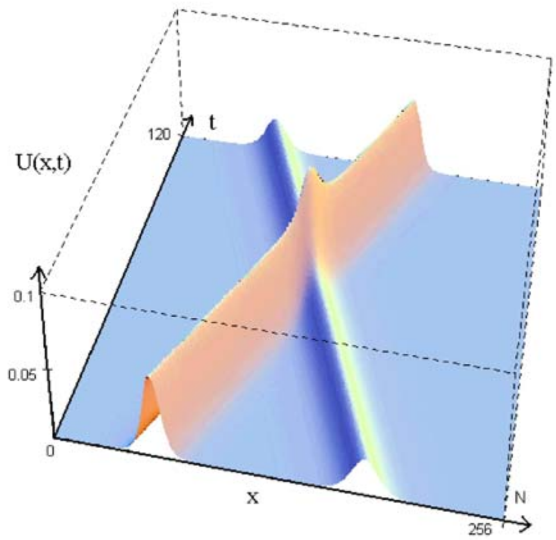

Figure 2. Two wave interaction.

4.1 Two wave interaction

As an example of a multiple solitary wave interaction we consider initially two solutions of the form of equation (3)

$$u_i(x,t) = 2b_i^2 sech^2\left(b_i(x - x_i - c_i t)\right), \quad i = 1,2 \tag{8}$$

with the initial condition

$$u(x,0) = \begin{cases} u_1(x,0) & \text{for} \quad [0, x_k] \\ u_2(x,0) & \text{for} \quad (x_k, x_N] \end{cases} \tag{9}$$

where x_N is the x corresponding to the Nth element (for $dx = 1$, $x_N \equiv N$), $b_1 = 0.15$, $b_2 = 0.1$, $x_1 = 40$, $x_2 = 180$ and $N = 256$. The interaction of the two solitons is shown in Figure2. We then examine the stability of this interaction. This can be done e.g. by fixing the value of b_1 and varying b_2 until it reaches a value where the wave blows up. The results are shown here in Figure 3.

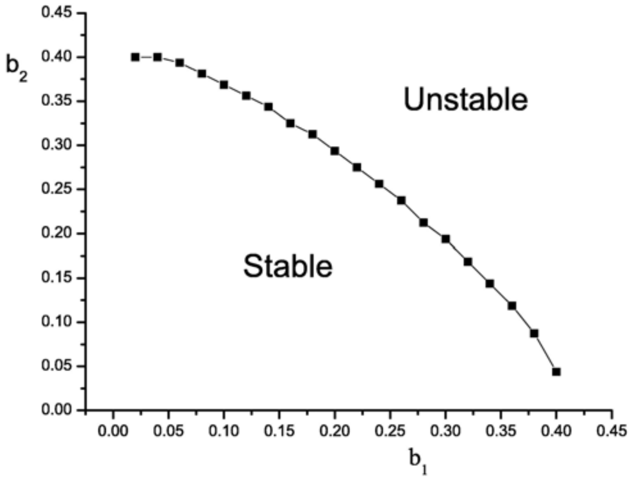

Figure 3. Stability region for two wave interaction.

5 CONCLUSIONS

In this paper we have applied an efficient numerical scheme for following the evolution of nonlinear solitary waves of the Boussinesq equation (1). The scheme consists of a combination of finite differences in time and a Fourier spectral analysis in space. We studied some simple cases i.e. the propagation of one soliton and the interaction of two such solutions moving in opposite directions. We also defined numerical bounds for the parameters of the wave that ensure the dynamical stability of the waves for very long time intervals. These results are promising, but clearly further studies are needed to extend these methods e.g. to the case of the Boussinesq equation in 2 spatial dimensions. In the future we plan to proceed with such studies, making also detailed comparisons with other numerical methods.

6 ACKNOWLEDGEMENTS

Ch. Skokos was partially supported by the Research Committee of the Academy of Athens. This work was carried out within the framework of the Education and Initial Vocational Training Program - Archimedes, Technological Educational Institution (T.E.I.) Athens, Archimedes project 'Computational Methods for Applied Technological Problems' funded 75% from E.U. and 25% from the Greek Government.

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